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Lexicographically optimal base of a submodular system with respect to a weight vector

by

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ABSTRACT

We show the existence of a lexicographically optimal base of a submodular system with respect to a weight vector. We also show a greedy procedure to get it through an algebraic consideration.

1. Introduction

Submodular system has been developed by S. Fujishige [1978–1987]. He posed an algorithm to get a lexicographically optimal base of a polymatroid with respect to a weight vector through geometric consideration [1980]. We have shown that the same results hold for a submodular system with $f(A) > 0(\emptyset \neq A \in \mathcal{D})$ and have presented a greedy procedure in an algebraic way [1987]. In response to our work and to questions proposed by the author, S. Fujishige [1987] has extended the same results for an arbitrary submodular system and has presented an algroithm to get it. His algorithm, which is not a direct extension of the algorithm for polymatroid, contains an oracle computation which has been pointed out by G. Morton, R. von Randow and K. Ringwald [1985]. Here, we show a greedy procedure to get it through algebraic consideration, which is quite different from Fujishige's algorithm [1980, 1987], but is an algebraic counterpart of his geometric consideration.

Submodular system is essentially a poset greedoid with submodular function on it, which is implicitly stated in S. Fujishige and N. Tomizawa [1983]. Greedoids are created and has been investigated by B. Korte and L. Lovász [1982–1986]. Our result is a natural consequence through the study of greedoids and submodular systems.

2. Submodular system, submodular polyhedra and their basic characteristics

We use the same symbol and terminology as that of S. Fujishige [1984]. Let E be a finite set and denote by 2^E the set of all the subsets of E. Let a collection $\mathcal{D}$ of subsets of E be a *distributive lattice* with set union and intersection as the lattice operations, i.e., for any $X, Y \in \mathcal{D}$ we have $X \cup Y, X \cap Y \in \mathcal{D}$. A function f from $\mathcal{D}$ to the set R of reals is called a *submodular function* on $\mathcal{D}$ if for each pair of $X, Y \in \mathcal{D}$

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y).$$

A pair $(\mathcal{D}, f)$ of a distributive lattice $\mathcal{D} \subseteq 2^E$ and a submodular function $f : \mathcal{D} \to R$ is called a *submodular system.* We assume that $\emptyset, E \in \mathcal{D}$ and $f(\emptyset) = 0$. Note that the value $f(\emptyset)$ doesn't affect the other value $f(A)$ at $A \in \mathcal{D}$ because $A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$. Given a submodular system $(\mathcal{D}, f)$, define a polyhedron P_f by

$$P_f := \{x \in R^E \mid x(X) \leq f(X)(\forall X \in \mathcal{D})\},$$

where R^E is the set of vectors $x = (x(e) : e \in E)$ with coordinates indexed by E and $x(e) \in R(e \in E)$ and

$$x(X) := \sum_{e \in X} x(e).$$

We call P_f the *submodular polyhedron* associated with the submodular system $(\mathcal{D}, f)$. Define

$$B_f := \{x \in P_f \mid x(E) = f(E)\},$$

which is called the *base polyhedron* associated with $(\mathcal{D}, f)$.

Lemma 2.1 *Let $x \in P_f$ and $A, B \in \mathcal{D}$. If $x(A) = f(A), x(B) = f(B)$, then $x(A \cap B) = f(A \cap B)$ and $x(A \cup B) = f(A \cup B)$ hold.*

Proof. Same as that of S. Fujishige [1978].

□

Let χ_u be a characteristic function of u, i.e., $\chi_u(e) = 1$ for $e = u$ and $\chi_u(e) = 0$ for $e \in E \backslash \{u\}$. Define a saturation function sat ():$P_f \to 2^E$ by $sat(x) := \{u \in E \mid \forall_{d>0}, x + d\chi_u \notin P_f\}(x \in P_f)$. Then we have the following lemma, where $\wp(x) := \{A \in \mathcal{D} \mid x(A) = f(A)\}$.

Lemma 2.2 *Let $x \in P_f$. Then sat(x) satisfies*

$$\mathrm{sat}(x) \in \mathcal{D}, x(\mathrm{sat}(x)) = f(\mathrm{sat}(x)).$$

Furthermore, $p(x)$ is a distributive lattice with a partial order relation defined by the set inclusion and sat(x) is the maximum element of $p(x)$.

Proof. Same as that of S. Fujishige [1980].

□

Note that sat(x) is a function from P_f into $\mathcal{D}$.

Lemma 2.3 *Let $x \in P_f$. Then $x \in B_f$ iff sat$(x) = E$.*

Proof. Use the definition of B_f and Lemma 2.2.

□

For $x \in P_f, u \in \mathrm{sat}(x)$, we can define *dependence function* dep(): $P_f \to \mathcal{D}$ and also we can introduce capacity, exchande capacity and so on (Fujishige [1984,1987]), but we don't go into the details because we don't use them.

Let $n := |E|$. For any real sequences $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ of length n, a is called *lexicographically greater than or equal to* b if for some $j \in \{1, \ldots, n\}$

$$\begin{aligned} &a_i = b_i \quad (i = 1, \ldots, j-1) \\ &a_j > b_j \\ &\text{or} \\ &a_i = b_i \quad (i = 1, \ldots, n). \end{aligned}$$

A vector $w \in R^E$ such that $w(e) > 0 (e \in E)$ is called a *weight vector.* For a vector $x \in R^E$, denote by $T(x)$ the n–tuple (or sequence) of the numbers $x(e)(e \in E)$ arranged in order of increasing magnitude. Given a weight vector w, a base x of $(\mathcal{D}, f)$ is called a *lexicographically optimal base with respect to the weight vector* w if the n–tuple $T((x(e)/w(e))_{e \in E})$ is lexicographically maximum among all n–tuples $T((y(e)/w(e))_{e \in E})$ for all bases y of $(\mathcal{D}, f)$. The mathematical Programming problem to get $x \in B_f$ such that

$$T((x(e)/w(e))_{e\in E}) = \underset{\textit{subject to } y \in B_f}{\textit{Lexicographically maximum}} \quad T((y(e)/w(e))_{e \in E})$$

is called wlob (weighted lexicographically optimal base) problem for submodular system.

3. Existence and uniqueness of a lexicographically optimal base with respect to a weight vector

Let $c_1 := min\{\frac{f(A)}{w(A)} \mid \emptyset \neq A \in \mathcal{D}\}, u_{c_1}(e) := c_1 w(e)(e \in E)$. Then we see that $u_{c_1} \in P_f$ holds. By Lemma 2.2, we have $u_{c_1}(\mathrm{sat}(u_{c_1})) = f(\mathrm{sat}(u_{c_1}))$. Let A_1 be a set such that $c_1 = \frac{f(A_1)}{w(A_1)}, \emptyset \neq A_1 \in \mathcal{D}$. Then $A_1 \subseteq \mathrm{sat}(u_{c_1})$, because $\forall e \in A_1, \forall d > 0, (u_{c_1} + d\chi_e)(A_1) = c_1 w(A_1) + d > f(A_1)$. Thus we get $\emptyset \neq \mathrm{sat}(u_{c_1}) \in \mathcal{D}$. Therefore, we are in a position such that

$$u_{c_1}(e) = c_1 w(e)(e \in E), u_{c_1} \in P_f, \emptyset \neq \mathrm{sat}(u_{c_1}) \in \mathcal{D} \text{ and } u_{c_1}(\mathrm{sat}(u_{c_1})) = f(\mathrm{sat}(u_{c_1})). \tag{3.1}$$

In case $\mathrm{sat}(u_{c_1}) = E$, by Lemma 2.3, we see that

$$u_{c_1} \in B_f. \text{ STOP}$$

In case $\mathrm{sat}(u_{c_1}) \subsetneqq E$, let $\epsilon_1 := min\{\frac{f(A)-u_{c_1}(A)}{w(A)\backslash\mathrm{sat}(u_{c_1}))} \mid A\backslash\mathrm{sat}(u_{c_1}) \neq \emptyset, A \in \mathcal{D}\}$. Then by Lemma 2.1, we get $\epsilon_1 > 0$. Let $c_2 := c_1 + \epsilon_1$, and let

$$u_{c_2}(e) := \begin{cases} c_1 w(e) = u_{c_1}(e) & \text{for } e \in \mathrm{sat}(u_{c_1}), \\ c_2 w(e) = u_{c_1}(e) + \epsilon_1 w(e) & \text{for } e \in E\backslash\mathrm{sat}(u_{c_1}). \end{cases}$$

By the definition of u_{c_2} and ϵ_1, and by the fact that $u_{c_1} \in P_f$, we get $u_{c_2} \in P_f$. Furthermore we get $p(u_{c_1}) \subseteq p(u_{c_2})$ and so $\mathrm{sat}(u_{c_1}) \subseteq \mathrm{sat}(u_{c_2})$. From the definition of ϵ_1, we have a set $A_1 \in \mathcal{D}, A_1\backslash\mathrm{sat}(u_{c_1}) \neq \emptyset$ such that $\epsilon_1 = \frac{f(A_1)-u_{c_1}(A_1)}{w(A_1\backslash\mathrm{sat}(u_{c_1}))}$. Then

$$\begin{aligned} u_{c_2}(A_1) &= u_{c_2}(A_1 \cap \mathrm{sat}(u_{c_1})) + u_{c_2}(A_1\backslash\mathrm{sat}(u_{c_1})) \\ &= c_1 w(A \cap \mathrm{sat}(u_{c_1})) + (c_1 + \epsilon_1)w(A_1\backslash\mathrm{sat}(u_{c_1}))[\text{by the definition of } u_{c_2}] \\ &= c_1 w(A_1) + \epsilon_1 w(A_1\backslash\mathrm{sat}(u_{c_1})) = u_{c_1}(A_1) + \epsilon_1 w(A_1\backslash\mathrm{sat}(u_{c_1})) = f(A_1) \end{aligned}$$

and so $A_1 \in p(u_{c_2})$.

By Lemma 2.1 and $\mathrm{sat}(u_{c_1}) \in p(u_{c_2})$, we have sat $(u_{c_1}) \subsetneqq^* \mathrm{sat}(u_{c_1}) \cup A \in p(u_{c_2})$. Thus $\mathrm{sat}(u_{c_1}) \subsetneqq \mathrm{sat}(u_{c_2})$. From Lemma 2.2 and $u_{c_2} \in P_f$, we have

$$u_{c_2}(\mathrm{sat}(u_{c_2})) = f(\mathrm{sat}(u_{c_2})). \tag{3.2}$$

Therefore, we are in a position such that

$$u_{c_2}(e) = \begin{cases} c_1 w(e)(e \in \mathrm{sat}(u_{c_1})) \\ c_2 w(e)(e \in E\backslash\mathrm{sat}(u_{c_1})), \end{cases} u_{c_i} \in P_f(i = 1,2), \emptyset \neq \mathrm{sat}(u_{c_1}) \subsetneqq \mathrm{sat}(u_{c_2}) \in \mathcal{D},$$

$$u_{c_i}(\mathrm{sat}(u_{c_i})) = f(\mathrm{sat}(u_{c_i}))(1 \leq i \leq 2) \text{ and } c_1 < c_2. \tag{3.3}$$

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* $X \subsetneqq Y$ means that X is a proper subset of Y.

Continuing this process, we get u_{c_p} such that $\mathrm{sat}(u_{c_p}) = E$, i.e., $u_{c_p} \in B_f$. Set

$$c(e) := \left\{ \begin{array}{l} c_1(e \in \mathrm{sat}(u_{c_1})) \\ c_2(e \in \mathrm{sat}(u_{c_2})\backslash\mathrm{sat}(u_{c_1})) \\ \vdots \\ c_i(e \in \mathrm{sat}(u_{c_i})\backslash\mathrm{sat}(u_{c_{i-1}})) \\ \vdots \\ c_p(e \in \mathrm{sat}(u_{c_p})\backslash\mathrm{sat}(u_{c_{p-1}}) = E\backslash\mathrm{sat}(u_{c_{p-1}}). \end{array} \right\} \tag{3.4}$$

Then we have

$$u_{c_p}(e) = \left\{ \begin{array}{l} c_1 w(e)(e \in \mathrm{sat}(u_{c_1})) \\ c_2 w(e)(e \in \mathrm{sat}(u_{c_2})\backslash\mathrm{sat}(u_{c_1})) \\ \vdots \\ c_i w(e)(e \in \mathrm{sat}(u_{c_i})\backslash\mathrm{sat}(u_{c_{i-1}})) \\ \vdots \\ c_p w(e)(e \in \mathrm{sat}(u_{c_p})\backslash\mathrm{sat}(u_{c_{p-1}})) \end{array} \right.$$

$u_{c_p} \in B_f, \emptyset \neq \mathrm{sat}(u_{c_1}) \not\subseteq \ldots \not\subseteq \mathrm{sat}(u_{c_p}) = E$ which are all in $\mathcal{D}, u_{c_i}(\mathrm{sat}(u_{c_i})) = f(\mathrm{sat}(u_{c_i}))(1 \leq i \leq p)$ and

$$c_1 < \ldots < c_p. \tag{3.5}$$

Note. For a *positive* submodular system $(\mathcal{D}, f)$, i.e., submodular system with $f(A) > 0(\emptyset \neq A \in \mathcal{D})$, we see that $c_1 > 0$.

Theorem 3.1 *(Existence) Let $c(e)(e \in E)$ be those defined by (3.4). Then the vector x defined by*

$$x = (c(e)w(e))_{e \in E} \tag{3.6}$$

is a lexicographically optimal base with respect to the weight vector w.

Proof. Let $z \in B_f$. We show that

$$T((z(e)/w(e))_{e \in E}) \overset{<}{l} T((x(e)/w(e))_{e \in E}) \tag{3.7}$$

holds. First note that

$$z(A) \leq f(A) \qquad (\emptyset \neq A \in \mathcal{D}) \tag{3.8}$$

holds. Let $q := (q_1, \ldots, q_n)$, $n = |E|$, be any permutation corresponding to x such that

$$\frac{x(q_1)}{w(q_1)} = \ldots = \frac{x(q_{j_1})}{w(q_{j_1})} = c_1 < \frac{x(q_{j_1+1})}{w(q_{j_1+1})} = \ldots = \frac{x(q_{j_2})}{w(q_{j_2})} = c_2 < \ldots <$$

$\frac{z(q_{j_{p-1}+1})}{w(q_{j_{p-1}+1})} = \ldots = \frac{z(q_{j_p})}{w(q_{j_p})} = c_p, j_p = n, c_{j_0} = 0$. Let $S_i = \{q_{j_{i-1}+1}, q_{j_{i-1}+2}, \ldots, q_{j_i}\}(1 \leq i \leq p)$. Then we have $S_1 = \mathrm{sat}(u_{c_1}), S_i = \mathrm{sat}(u_{c_i})\backslash\mathrm{sat}(u_{c_{i-1}})(2 \leq i \leq p)$.

If $\frac{z(q_1)}{w(q_1)} < c_1$, then (3.7) holds.

If $\frac{z(q_1)}{w(q_1)} \geq c_1$, $\frac{z(q_2)}{w(q_2)} < c_1$, the (3.7) holds.

$\vdots$

If $\frac{z(q_1)}{w(q_1)} \geq c_1, \ldots, \frac{z(q_{j_1})}{w(q_{j_1})} \geq c_1$, then we see that

$$\frac{z(e)}{w(e)} = \frac{x(e)}{w(e)} = c_1(e \in S_1) \tag{3.9}$$

holds by $z(S_1) \geq c_1 w(S_1) = u_{c_1}(S_1) = f(S_1)$ and by (3.8).

If $\frac{z(e)}{w(e)} = c_1(e \in S_1)$; $\frac{z(q_{j_1+1})}{w(q_{j_1+1})} < c_2$, then (3.7) holds.

If $\frac{z(e)}{w(e)} = c_1(e \in S_1)$, $\frac{z(q_{j_1+1})}{w(q_{j_1+1})} \geq c_2$, $\frac{z(q_{j_1+2})}{w(q_{j_1+2})} < c_2$, then (3.7) holds.

$\vdots$

If $\frac{z(e)}{w(e)} = c_1(e \in S_1)$, $\frac{z(q_{j_1+1})}{w(q_{j_1+1})} \geq c_2, \ldots, \frac{z(q_{j_2})}{w(q_{j_2})} \geq c_2$, then we see that $\frac{z(e)}{w(e)} = c_2 = \frac{x(e)}{w(e)}(e \in S_2)$ holds because $z(e) = c_1 w(e)(e \in S_1)$ and $z(S_2 + S_1) \leq f(S_2+S_1), f(S_2+S_1) = u_{c_2}(S_2+S_1) = z(S_1)+c_2 w(S_2) \leq z(S_2+S_1)$. Continuing in this way, we see that (3.7) holds for any $z \in B_f$.

□

Theorem 3.2 *(Uniqueness, Fujishige, S. [1980]) Let $c(e)(e \in E)$ be those defined by (3.4). Then the vector x defined by (3.6) is the unique lexicographically optimal base of $(\mathcal{D}, f)$ with respect to a weight vector w.*

Proof. Same as that of Fujishige, S. [1980]. Use (3.5), especially $\mathrm{sat}(u_{c_i}) \in \mathcal{D}$, $u_{c_i}(\mathrm{sat}(u_{c_i})) = f(\mathrm{sat}(u_{c_i}))$.

□

Based on these algebraic arguments, we present an algorithm to get the lexicographically optimal base of a submodular system $(\mathcal{D}, f)$ with respect to a weight vector w.

Algorithm to get the lexicographically optimal base

Step 1. <u>Set</u> $i := 1$ and <u>compute</u> $c_i := min\{\frac{f(A)}{w(A)} \mid \emptyset \neq A \in \mathcal{D}\}$ and

<u>set</u> $u_{c_i}(e) := c_i w(e)(e \in E)$.

Step 2. <u>If</u> $\mathrm{sat}(u_{c_i}) = E$, <u>then STOP</u>.

Step 3. Compute $\epsilon_i := min\{\frac{f(A)-u_{c_i}(A)}{w(A\backslash \mathrm{sat}(u_{c_i}))} \mid A \in \mathcal{D}, A\backslash \mathrm{sat}(u_{c_i}) \neq \emptyset\}$

and set $c_{i+1} := c_i + \epsilon_i$ and set

$$u_{c_{i+1}}(e) := \begin{cases} u_{c_i}(e) & \text{for } e \in \mathrm{sat}(u_{c_i}) \\ u_{c_i}(e) + \epsilon_i w(e) & \text{for } e \in E\backslash \mathrm{sat}(u_{c_i}). \end{cases}$$

Set $i := i + 1$ and go to Step 2.

Theorem 3.3 *(Fujishige, S. [1980]) Let $\hat{x} \in B_f$ and let w be a weight vector. Define*

$$\hat{c}(e) := \hat{x}(e)/w(e)(e \in E)$$

and let the distinct numbers of $\hat{c}(e)(e \in E)$ be given by

$$\hat{c_1} < \hat{c_2} < \ldots < \hat{c}_{\hat{p}}.$$

Furthermore, define $\hat{S}_i \subseteq E(1 \leq i \leq \hat{p})$ by $\hat{S}_i := \{e \in E \mid \hat{c}(e) \leq \hat{c}_i\}(1 \leq i \leq \hat{p})$. Then the following three conditions are equivalent:

(i) $\hat{x}$ *is the lexicographically optimal base of P_f with respect to w;*

(ii) $\hat{S}_i \in \mathcal{D}$ and $\hat{x}(\hat{S}_i) = f(\hat{S}_i)(1 \leq i \leq \hat{p})$;

(iii) *For any $e \in \hat{S}_i$, $\emptyset \neq \mathrm{dep}(\hat{x}, e) \subseteq \hat{S}_i(1 \leq i \leq \hat{p})$.*

Remark If one of the three conditions holds, then we have $\hat{p} = p$.

Given a submodular system $(\mathcal{D}, f)$ and a weight vector w and $p > 1$, define a mathematical programming problem

$$\mathrm{P} \;: \text{minimize } f_w(x) = \frac{1}{p}\sum_{e\in E}\frac{x(e)^p}{w(e)^{p-1}} \text{ subject to } x \in B_f \text{ and } x \geq 0.$$

Fujishige, S. [1980]showed that for a polymatroid $(\mathcal{D}, f)$ with $p = 2$, its unique solution is the lexicograhphically optimal base w.r.t. w. Morton, G. and von Randow, R. and Ringwald, K. [1985] extended it for $p > 1$, where $(\mathcal{D}, f)$ is a polymatroid. We can easily see that for a positive submodular system $(\mathcal{D}, f)$ with $p > 1$, the same result holds. As for an arbitrary submodular system, P might be infeasible. For example, for a submodular system $(\mathcal{D}, f)$ with $f(A) < 0$ $(A \in \mathcal{D})$. So, consider another problem

$$\hat{P} : \text{minimize } f_w(x) = \frac{1}{p}\sum_{e\in E}\frac{x(e)^p}{w(e)^{p-1}} \text{ subject to } x \in B_f.$$

We have an example for which $\hat{P}$ has no optimal solution as follows: Let $E = \{1,2,3\}, \mathcal{D} = \{\emptyset, \{3\}, \{1,2,3\}\}, f(\emptyset) = 0, f(\{3\}) = -2, f(\{1,2,3\}) = -3$. Then $(\mathcal{D}, f)$ is a submodular system with base polyhedron $B_f = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = -3, x_3 \leq -2\}$. Let $w = (1,1,1)$. The lexicographically optimal base x^* becomes $x^* = (-\frac{1}{2}, -\frac{1}{2}, -2)$. Let $p = 3$ and let $x_1 = x_2 = -\frac{(t+3)}{2}, x_3 = t(\leq -2)$. Then $(x_1, x_2, x_3) \in B_f$ with $3f_w(x) = t^3 - \frac{1}{4}(t+3)^3 \to -\infty$ as $t \to \infty$. Problem $\hat{P}$ for this case has no minimum solution. For an even natural number p, if there exists a minimum solution $\hat{x}$ for $\hat{P}$, then we see that $\hat{x}$ is the lexicographically optimal base w.r.t. w.

Theorem 3.4 *(Fujishige, S. [1980], Morton, G. and von Randow, R. and Ringwald, K. [1985])*
Let x^* be the lexicographically optimal base of a positive submodular system $(\mathcal{D}, f)$ with respect to a weight vector w and let $p > 1$. Then x^* is the unique optimal solution of the problem P.

4. Example

We will show here that the first problem of G. Morton, R. von Randow and K. Ringwald [1985]can be solved within our framework. Their problem is as follows:

$$\min \sum_{j=1}^{n} \lambda_j x_j^p \text{ subject to } Ax \geq c,\ x \geq 0, \tag{4.1}$$

where $\lambda_j > 0(1 \leq j \leq n), p > 1, c_n \geq c_{n-1} \geq \ldots \geq c_1 \geq 0$, and

$$A = (a_{ij})_{n \times n} \text{ with } a_{ij} = \begin{cases} 1, & i \geq j, \\ 0, & i < j. \end{cases}$$

Let e_i be the i-th column vector of A, $E := \{e_i \mid 1 \leq i \leq n\}, F_j := \{e_i \mid 1 \leq i \leq j\}(1 \leq j \leq n), F_0 := \emptyset, D_j := E \backslash F_j = \{e_{j+1}, \ldots, e_n\}(0 \leq j \leq n)$. Let $\mathcal{D} = \{E = D_0, D_1, \ldots, D_{n-1}, D_n = \emptyset\}$. Let $\rho(D_j) := c_n - c_j(0 \leq j \leq n)$, where $c_0 = 0$. Then $(E, \mathcal{D}, \rho)$ is a submodular system with $\emptyset, E \in \mathcal{D}, \rho(\emptyset) = 0$. For $x, y \in \mathbb{R}_+^n$, define $x \leq y$ if $x(e) \leq y(e)(e \in E)$, where $\mathbb{R}_+$ is the set of nonnegative reals. $(\mathbb{R}_+^n, \leq)$ is a poset with this partial order. Define $P := \{x \in \mathbb{R}_+^n \mid Ax \geq c\}, O(4.1) :=$ the set of optimal solutions to (4.1), minimal $P :=$ the set of minimal elements of P. Then we easily see that

$$O(4.1) \subseteq B_\rho \subseteq \text{minimal } P \subseteq P,$$

Hence problem (4.1) is equivalent to

$$\min\{\frac{1}{p}\sum_{i=1}^{n} x(e_i)^p w(e_i)^{1-p} \mid x \in B_\rho\},$$

where $w(e_i) = \lambda_i^{-\frac{1}{(p-1)}}$. Let $d_j = \sum_{i=1}^{j} w(e_i) (1 \le j \le n)$ and $d_0 = 0$. Then $w(D_j) = d_n - d_j (0 \le j \le n)$. Apply our algorithm to this problem:

$$c_1' := \min\{\frac{\rho(D_j)}{w(D_j)} \mid 0 \le j \le n-1\} = \min\{\frac{c_n - c_0}{d_n - d_0}, \frac{c_n - c_1}{d_n - d_1}, \frac{c_n - c_2}{d_n - d_2}, \dots, \frac{c_n - c_{n-1}}{d_n - d_{n-1}}\}.$$

Let $s'(0) = n$ and $c_1' = \frac{c_n - c_{s'(1)}}{d_n - d_{s'(1)}}$ and $u_{c_1'}(e_i) = c_1' w(e_i) (1 \le i \le n)$. Then $u_{c_1'}(D_j) = c_1'(d_n - d_j)$, $\mathrm{sat}(u_{c_1'}) = \bigcup\{A \mid A \in \mathcal{D}, u_{c_1'}(A) = \rho(A)\} = D_{s'(1)}$ for which $s'(1)$ is the least index j such that $c_1' = \frac{c_n - c_j}{d_n - d_j}$, $0 \le s'(1) < s'(0)$.
If $s'(1) = 0$, then $\mathrm{sat}(u_{c_1'}) = E$. STOP.
If $s'(1) \ne 0$, then $\mathrm{sat}(u_{c_1'}) \ne E$ and so compute

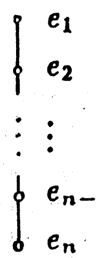
$$\epsilon_1' := \min\{\frac{\rho(A) - u_{c_1'}(A)}{w(A \backslash \mathrm{sat}(u_{c_1'}))} \mid A \in \mathcal{D}, A \backslash \mathrm{sat}(u_{c_1'}) \ne \emptyset\} = \min\{\frac{c_n - c_j - c_1'(d_n - d_j)}{d_{s'(1)} - d_j} \mid$$

$$0 \le j \le n-1, j < s'(1)\}, \text{ where } \frac{c_n - c_j - c_1'(d_n - d_j)}{d_{s'(1)} - d_j} = \frac{c_{s'(1)} - c_j}{d_{s'(1)} - d_j} - c_1'.$$

Let $\epsilon_1' := \frac{c_{s'(1)} - c_{s'(2)}}{d_{s'(1)} - d_{s'(2)}} - c_1'$. Then $(d_{s'(2)}, c_{s'(2)})$ is a point (d_j, c_j), $0 \le j < s'(1)$ with the smallest slope coefficient $\frac{c_{s'(1)} - c_j}{d_{s'(1)} - d_j}$. Hence we see that

$$s'(0) = n = s(m), s'(1) = s(m-1), \dots, s'(m-1) = s(1), s'(m) = s(0),$$

which is the same result as that of G. Morton, R. von Randow and K. Ringwald, although the decision proceeds inversely. The reader would have noticed that the $(E, \mathcal{D})$ here, is a poset greedoid which comes from a chain as follows:



The reason for the inverse decision process will be investigated in another paper.

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